An overview of

Counting Rules in Statistics and Computer Science

Introduction

This paper is written for students at Canada College who are taking Statistics or Discrete Math for Computer Science. Counting rules are taught in these classes.

Counting rules are used for computing probabilities, such as the number of possible combinations for a lock, how many routes can a package delivery driver take to deliver some number of packages, combinations from a restaurant menu, permutations of medical tests, or grouping heterogeneous items.

Most Statistics books present counting rules in five categories:

1. Fundamental Counting Rule
2. Factorial Rule
3. Permutations Rule, with repetition
4. Permutations Rule, without repetition
5. Combinations Rule

This paper follows the sequence of presentation used in Statistics text books written by Mario Triola. He is the author of text books used in Statistics classes at Canada College.

Overview

How the counting rules are used depends on the problem to solve. Example use of the rules are:

- **Fundamental counting rule**, to estimate possibilities for a combination lock or lottery ticket. Typically the formula is m^n

- **Factorial rule**, to count how many routes a package delivery driver can take. Typically this formula is n-factorial or n!

- **Permutations with repetition**, to count how many possible sequences exist for a given collection of items. Typically the formula is n!/(n(1)! * … n(k)!))

- **Permutations without repetition**, to count ways to sequence some items from a larger group of available items. This is commonly referred to as nPr with a formula of n!/(n-r)!

- **Combinations**, when items are selected without repetition and order does not matter, such as selecting three students from a class of 28. This is commonly referred to as nCr with a formula if n!/(r!(n-r))!

The challenge with counting rules is in determining which rule applies and therefore which formula to use. This is especially difficult with combinations and permutations. This paper is intended to help with that decision by enumerating all criteria for using each rule.

The Fundamental Counting and Factorial rules are probably the easiest to understand and to know when to use. They are explained first for that reason. Determining which Permutation or Combination rule to use requires evaluating several criteria:

- Are duplicates allowed or excluded?
- Is the selection with or without replacement?
- Does the sequence items make a difference?

Permutations and Combinations: Does order matter?

- Permutations of items are arrangements in which *different sequences of the same items **are** counted separately*. 3, 2, 1 would be considered different from 1, 2, 3.
- Combinations of items are arrangements in which *different sequences of the same item are **not** counted separately*. 3, 2, 1 and 1, 2, 3 would be considered the same.

Fundamental Counting Rule: m^n

The fundamental counting rule is used with problems such a how many different sequences of numbers are possible in a combination lock, a lottery ticket, or a PIN.

For example:

- A combination lock with five numbers, each number between 0 and 9, for a total of ten choices per number, has 10^4 possible "combinations". Each of the four positions can be one of ten values and there are four positions. Therefore the total number of possibilities is 10 * 10 * 10 * 10 or 10^4. Statisticians prefer the term "permutation lock" rather than "combination lock".

- A four-digit PIN, with each digit having a value from 1 to 6 would have 6 * 6 * 6 * 6 or 6^4 possible different values.

- A 12 digit lottery number having values from 1 to 16 would have 16^12 different possible values.

- Take four different letters and arrange them into a sequence of eight consecutive letters. Each position has four possibilities. There are 8 positions, so there are 4^8 = 65,536 possible sequences.

A variation of Fundamental Counting is used to tally arrangement of items with inconsistent possibilities, such as:

- how many ways can three varieties of wine be served with five different entrees and two different salads

- How many ways can four different car body styes, with ten different colors, and three different engines be arranged

The formula in these cases is m * n * o * p … where m … p represent the number of possible items for a given position. For example:

- Patients at a clinic have blood types A, B, AB, or O; and are categorized as having blood pressure that is high, low, or normal. There are 4 * 3 = 12 possible arrangements of blood type and blood pressure.

- Clinicians want to test 12 different medications on three different mice. How many ways are there to choose one mouse and one medication? m * n = 12 * 3 = 36

- A car is offered with four body styles, three engine sizes, and ten colors, for a total of 4 * 3 * 10 = 120 different possible arrangements.

- A restaurant offers three varieties of wine and with five different entrees. There are 3 * 5 = 15 different possible combinations of wine and entree.

- A sailboat can be configured with five different head sails, four different main sails, two different engines, and with or without a dodger. There are 5 * 4 * 2 * 2 different possible arrangements.

Factorial Rule: n!

This rule is so named because factorials are used to count possibilities:

- A delivery truck with 126 packages for 126 different addresses has 126 * 125 * 124…* 1 or 126! different possible routes. 126! is approximately 2.4 times ten to the 211th power.

- A presidential candidate wants to visit all 50 states. There are 50!, or approximately 3 times ten to the 64th possible routes to take.

The Factorial rule applies when all items under consideration are selected, as opposed to selecting a subset of items from a larger pool.

Birthday riddle

The birthday riddle is a classic problem in statistics. It is solved using complementary logic and the factorial rule.

The question usually starts with a given of 23 people in a room and the question is what is the probability that at least two people have the same birth day, as in the same month and day.

In statistics, when a problem includes computing an “at least” probability, this is an indicator that complementary probability logic may be the best approach. In this case, the complement would be the probability that all 23 people have a different birthday, which is easier to compute:

- Probability that person #1 has a unique birthday is 365/365
- Probability that person #2 has a unique birthday is 364/365
- Probability that person #3 has a unique birthday is 363/365

…

- Probability that person #23 has a unique birthday is 343/365

This means the probability that all 23 people have a unique birthday is (365*364*363*…343/365^23, which can be expressed more concisely as (365!/342!)/365^23. This is approximately equal to 0.493. The complement is 0.507, meaning there is a 50.7 percent chance that two people among 23 share the same birthday (mm/dd).

Permutations with repetition: n!/(n1!*n2!…n(k)!)

The formula for this rule is n!/(n1!*n2!…n(k)!)

In this case:

- **Order matters.** Different arrangements of the same items are counted separately. 3, 2, 1 is counted separately from 1, 2, 3.
- Select item **without replacement**

- **With repetition:** Some items are identical

For example:

- Given ten letters (a, a, a, a, b, b, c, c, d, e) how many different ways can these letters be arranged in sequence: n!/(n1!*n2!… n(k)!) or 10!/(4!*2!*2!*1!*1!) = 37,800

- How many ways can the letters in committee be arranged? There are nine letters, with three pairs and three singles = 9!/(2! *2!*2!*1!*1!*1!) = 45,360 sequences

- How many ways can the letters in Mississippi be arranged? There are 4{s}, 4{i}, 2{p}, and 1{M} = 11!/(4!*4!*2!*1!) = 34,650 sequences

- How many ways can the letters in Statistics be arranged? There are 3(s), 3(t), 2(i), 1(a), 1(c) = 10!/(3!*3!*2!*1!*1!) = 50,400 sequences.

Permutations without repetition: n!/(n-r)!

The formula for this rule is n!/(n-r)! where n is the number of available items and r is the number of selected items.

Common notation for this rule is nPr.

Use this rule when a subset of available items is selected. This rule is a variation of the factorial rule, but what is different is that not all items are selected and used.

When to use this rule:

- **Without repetition** means that all selected items are different.

- With this formula **order matters**. This means that rearrangements of the same items in different order are a different sequence. For example 3, 2, 1 is different from 1, 2, 3.

For example

- From a pool of 52 people choose a president, vice president, secretary, and treasurer. A person selected can only occupy one position. The number of ways to select four different people from a pool of 52 is n!/(n-r)! or 51!/(52-4)! or 6,497,400

- From a pool of five people choose four. The number of possible arrangements is 5!/(5-4)! = 5! = 120.

- If ten horses run a race the first three can be arranged in 10!/7! = 10*9*8 = 720 different ways.

- If a candidate wants to travel to 4 of 50 states there are 50!/46! possible itineraries.

- A lottery number with 5 different numbers each between 1 and 24 has 24!/19! possible sequences

Combinations Rule: n!/(r!*(n-r)!)

The formula for this counting rule is n!/(r!*(n-r)!)

Common notation for this rule is nCr.

When to use this rule:

- Count the number of ways to do something
- Count the total of n different items available
- Select r out of n number of items
- **Order does not matter**, meaning rearrangements of the same items are considered the same grouping. For example, 1, 2, 3 is considered the same as 3, 2, 1.
- Select **without repetition**

For example:

- Select six numbers from one to 38 without repetition = 38!/(6! *(38-6)!) = 38!/(6!*32*) = 2.760,681 different possible combinations.
- Select seven different subjects from a pool of 47 = 47!/(7!*(40!)) = 62,891,499

- Select three officers from a candidate pool of 11 = 11!/(3!*8!) = 165 possible combinations of three people.

- Select five cards from a deck of 52. There are 52!/(5!*47!) = 2,598,960 possible hands of five cards from a deck of 52.

Summary and conclusions.

These five rules are grouped to be applied to different counting scenarios. Students can expect to see problems with combinations in the workplace. Understanding these formulas and when to use them is a valuable skill.

Each rule is presented with details about when to use it, under what circumstances, and how to evaluate whether it is appropriate for the problem at hand. The examples are intended to make it easier to identify the appropriate rule.